Stable Evaluation of Influence Coefficients for Three-Dimensional Extended Boundary-Node Method

Taku Itoh[1], Ayumu Saitoh[2], and Atsushi Kamitani[3]
[1]Seikei University, 3-3-1, Kichijoji-kitamachi, Musashino, Tokyo, 180-8633, Japan, E-mail: taku@st.seikei.ac.jp
[2]University of Hyogo, 2167, Shosha, Himeji, Hyogo, 671-2280, Japan, E-mail: saitoh@eng.u-hyogo.ac.jp
[3]Yamagata University, 4-3-16, Johnan, Yonezawa, Yamagata, 992-8510, Japan, E-mail: kamitani@yz.yamagata-u.ac.jp

Abstract — For the purpose of stabilization of the three-dimensional eXtended Boundary Node Method (X-BNM), a new algorithm for evaluating influence coefficients has been proposed. Numerical experiments show that although the conventional algorithm cannot be employed in problems with a relatively small number of nodes, the proposed one is applicable to those with any number of nodes. However, the computational cost of the proposed algorithm slightly increases as compared with that of the conventional one. In addition, the proposed algorithm shows almost the same accuracy as the conventional one.

I. Introduction

Domain-type meshless methods such as Element-Free Galerkin Method (EFGM) have been applied to electromagnetic problems and have produced a lot of attractive results [1], [2]. Alternatively, boundary-type meshless methods such as Boundary-Node Method (BNM) [3] and Boundary Radial Point Interpolation Method (BRPIM) have also been applied to electromagnetic problems [4]. In BNM and BRPIM, the boundary surface must be divided into a set of integration cells to evaluate surface integrals. In this sense, these methods still have a concept of boundary elements partly.

To remove the integration cells completely, three-dimensional (3D) BNM has been recently reformulated. The reformulated method is called the eXtended Boundary-Node Method (X-BNM) [5]. In this method, the solution of nonlinear systems is indispensable for evaluating influence coefficients. However, solutions of the nonlinear systems sometimes do not converge to appropriate ranges for the case where an integral domain is a complicated shape. The purpose of this study is to propose a new algorithm for stable evaluation of influence coefficients for the X-BNM.

II. Influence Coefficients

If the X-BNM is applied to a 3D Laplace problem, the influence coefficients, G_{ij} and $H_{ij} (i, j = 1, 2, \ldots, N)$, can be written as

$$G_{ij} \equiv \int_{S_j} w^*(\boldsymbol{x}, \boldsymbol{x}_i)\phi_j(\boldsymbol{x}) \, \mathrm{d}S(\boldsymbol{x}), \tag{1}$$

$$H_{ij} \equiv \int_{S_j} \frac{\partial w^*}{\partial n}(\boldsymbol{x}, \boldsymbol{x}_i)\phi_j(\boldsymbol{x}) \, \mathrm{d}S(\boldsymbol{x}) + \frac{\phi_j(\boldsymbol{x}_i)\Omega_i}{4\pi}, \tag{2}$$

where $w^*(\boldsymbol{x}, \boldsymbol{x}_i) \equiv (4\pi|\boldsymbol{x} - \boldsymbol{x}_i|)^{-1}$, and S_j is a part of the boundary surface ∂V contained in a sphere of radius R and center $\boldsymbol{x}_j$. In addition, $\phi_j(\boldsymbol{x})$ denotes a shape function corresponding to the j-th boundary node $\boldsymbol{x}_j$ $(j = 1, 2, \ldots, N)$. Here, N is the number of boundary nodes and Ω_i is a solid angle on $\boldsymbol{x}_i$.

III. Evaluation of Influence Coefficients

In the X-BNM, a boundary surface is assumed as an implicit surface $f(\boldsymbol{x}) = 0$ and the shape function is assumed to have a support of radius R. Under the assumption, influence coefficients can be written in the form,

$$I = \int_S F \, \mathrm{d}S \ . \tag{3}$$

Here, S denotes a part of the implicit surface Π contained in a sphere of radius R and center $\boldsymbol{y}$. Different coordinates are used for the numerical integration of (3), depending on whether S contains a singularity $\boldsymbol{z}$ of $F(\boldsymbol{x})$ or not [5].

For the case where S contains a singularity $\boldsymbol{z}$, we use the 3D polar coordinate $(\rho^*, \theta^*, \varphi^*)$ whose origin coincides with the singularity $\boldsymbol{z}$. In addition, we employ a local Cartesian coordinate system $\langle \boldsymbol{z} : \boldsymbol{e}_x^*, \boldsymbol{e}_y^*, \boldsymbol{e}_z^* \rangle$ illustrated in Fig. 1(a). By using the system, arbitrary points $\boldsymbol{x}^*$ are expressed as $\boldsymbol{x}^* = \boldsymbol{z} + \rho^*(\sin\theta^*\cos\varphi^*\boldsymbol{e}_x^* + \sin\theta^*\sin\varphi^*\boldsymbol{e}_y^* + \cos\theta^*\boldsymbol{e}_z^*) \equiv \boldsymbol{g}^*(\rho^*, \theta^*, \varphi^*)$. Note that, on S, θ^* is a function of ρ^* and φ^*, i.e., $\theta^* = \theta^*(\rho^*, \varphi^*)$. This can be easily proved by using the implicit function theorem. The function $\theta^*(\rho^*, \varphi^*)$ is determined by solving a nonlinear equation, $f(\boldsymbol{g}^*(\rho^*, \theta^*, \varphi^*)) = 0$. The vector equation in the integral domain S is given by $\boldsymbol{x}^* = \boldsymbol{g}^*(\rho^*, \theta^*(\rho^*, \varphi^*), \varphi^*)$ $(0 \le \rho^* \le R^*(\varphi^*), 0 \le \varphi^* < 2\pi)$. Using the vector equation, the integration (3) can be rewritten as follows:

$$I = \int_0^{2\pi} \mathrm{d}\varphi^* \int_0^{R^*(\varphi^*)} \mathrm{d}\rho^* G^*(\rho^*, \varphi^*), \tag{4}$$

where

$$G^*(\rho^*, \varphi^*) \equiv \rho^* F(\boldsymbol{g}^*) \sqrt{[(\theta^*_{\rho^*}\rho^*)^2 + 1]\sin^2\theta^* + (\theta^*_{\varphi^*})^2}. \tag{5}$$

Note that the equation $\rho^* = R^*(\varphi^*)$ representing the edge of S is determined by solving the following nonlinear systems:

$$\sigma_1^*(\rho^*, \theta^*) \equiv f(\boldsymbol{g}^*(\rho^*, \theta^*, \varphi^*)) = 0, \tag{6}$$

$$\sigma_2^*(\rho^*, \theta^*) \equiv |\boldsymbol{g}^*(\rho^*, \theta^*, \varphi^*) - \boldsymbol{y}|^2 - R^2 = 0. \tag{7}$$

In the numerical evaluation of (4), the trapezoid formula and the Gauss-Legendre quadrature are applied to the φ^*- and ρ^*-directions, respectively. Throughout this paper, N_t and N_g denote the number of integration points for the trapezoid formula and that for the Gauss-Legendre quadrature, respectively. For solving the nonlinear systems (6) and (7), the Newton method is adopted. The concrete procedures for solving (6) and (7) are shown in the following pseudo code. In this code, ρ_m^* and φ_m^* are defined by $\rho_m^* = R^*(\varphi_m^*)$ and $\varphi_m^* = (m-1)\Delta\varphi^*$ $(m = 1, 2, \ldots, N_\mathrm{t})$ where $\Delta\varphi^* = 2\pi/N_\mathrm{t}$.

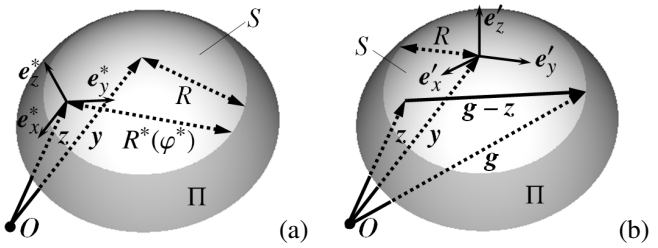

Fig. 1. (a) A local Cartesian coordinate system for the case where S contains a singularity z of $F(\boldsymbol{x})$. (b) A local Cartesian coordinate system for the proposed algorithm for determination of $R^*(\varphi^*)$.

Initial solutions (ρ_0^*, θ_0^*) are set;
for$(m = 1;\ m \le N_{\rm t};\ ++m)$\{
  (ρ_m^*, θ_m^*) = Newton2D$(\rho_{m-1}^*, \theta_{m-1}^*, \varphi_m^*, \Delta\varphi^*)$;
\}
Newton2D$(\rho_{\rm ini}^*,\ \theta_{\rm ini}^*,\ \varphi_{\rm const}^*,\ \Delta\varphi^*)$\{
  (ρ^*, θ^*) are determined by solving (6) and (7) with
  initial solutions $(\rho_{\rm ini}^*, \theta_{\rm ini}^*)$ on the assumption $\varphi^*=\varphi_{\rm const}^*$;
  if$((\rho^* < 0 \parallel \rho^* > 2R) \parallel (\theta^* < 0 \parallel \theta^* > \pi))$\{
    $\hat{\varphi}^* = \varphi_{\rm const}^* - \Delta\varphi^*/2$;
    (ρ^*, θ^*) = Newton2D$(\rho_{\rm ini}^*,\ \theta_{\rm ini}^*,\ \hat{\varphi}^*,\ \Delta\varphi^*/2)$;
  \}
  return (ρ^*, θ^*);
\}

IV. STABILIZATION

In the above pseudo code, the function "Newton2D" is recursively called until solutions (ρ^*, θ^*) simultaneously converge to appropriate ranges that are $0 \le \rho^* \le 2R$, $0 \le \theta^* \le \pi$. Unfortunately, for the case where an integral domain S is a complicated shape, the recursive process is not finished. For this case, this process is permanently repeated because the approximate solutions (ρ^*, θ^*) fall into inappropriate ranges.

For the purpose of stable evaluation of influence coefficients, we propose a new algorithm for determining $\rho^* = R^*(\varphi^*)$. In this algorithm, we employ a local Cartesian coordinate system $\langle \boldsymbol{y} : \boldsymbol{e}'_x, \boldsymbol{e}'_y, \boldsymbol{e}'_z \rangle$ illustrated in Fig. 1(b). By using the coordinate system, arbitrary points $\boldsymbol{x}$ are expressed as $\boldsymbol{x} = \boldsymbol{y} + \rho(\sin\theta\cos\varphi\ \boldsymbol{e}'_x + \sin\theta\sin\varphi\ \boldsymbol{e}'_y + \cos\theta\ \boldsymbol{e}'_z) \equiv \boldsymbol{g}(\rho, \theta, \varphi)$. In this coordinate system, a length between $\boldsymbol{y}$ and the edge of S is always R. On the edge of S, θ can be determined by solving a nonlinear system $f(\boldsymbol{g}(R, \theta, \varphi)) = 0$ with φ = constant. Therefore we obtain $R^*(\varphi^*) = |\boldsymbol{g}(R, \theta, \varphi) - \boldsymbol{z}|$ (see Fig. 1(b)). Note that φ^* is not determined. For determining φ^*, $\boldsymbol{x} = \boldsymbol{g}(R, \theta, \varphi)$ is represented in the coordinate system $\langle \boldsymbol{z} : \boldsymbol{e}^*_x, \boldsymbol{e}^*_y, \boldsymbol{e}^*_z \rangle$ as $\boldsymbol{x}^* = (\boldsymbol{e}^*_x \otimes \boldsymbol{e}^*_x + \boldsymbol{e}^*_y \otimes \boldsymbol{e}^*_y + \boldsymbol{e}^*_z \otimes \boldsymbol{e}^*_z) \cdot (\boldsymbol{x} - \boldsymbol{z})$. By using the components of $\boldsymbol{x}^*$, φ^* is determined as $\varphi^* = \tan^{-1}(y^*/x^*)$ $(0 \le \varphi^* < 2\pi)$, where $x^* \equiv (\boldsymbol{x} - \boldsymbol{z}) \cdot \boldsymbol{e}^*_x$ and $y^* \equiv (\boldsymbol{x} - \boldsymbol{z}) \cdot \boldsymbol{e}^*_y$.

V. NUMERICAL EXPERIMENTS

In this section, the performance of the proposed algorithm is compared with that of the conventional algorithm. To this end, the X-BNM with the proposed algorithm (new X-BNM) and the X-BNM with the conventional algorithm (conventional X-BNM) are applied to a 3D Laplace problem. A boundary shape is assumed as $f(\boldsymbol{x}) = x^2 + y^2/4 + z^2/36 - 1 = 0$. The boundary condition is chosen so that the analytic solution may be $u = 2\bar{r}^3 P_3^1(\cos\bar{\theta})\cos\bar{\varphi}$. Here, $(\bar{r}, \bar{\theta}, \bar{\varphi})$ is a usual 3D polar coordinates and $P_3^1(x)$ is the associated Legendre function. In addition, Dirichlet and Neumann conditions are assumed on given boundary nodes $\boldsymbol{x}_k$ with $z_k \ge 0$ and those with $z_k < 0$, respectively. Moreover, the numbers of integration points are assumed as follows: $N_{\rm t} = 11, N_{\rm g} = 5$ for $z \notin S$ and $N_{\rm t} = 55, N_{\rm g} = 5$ for $z \in S$.

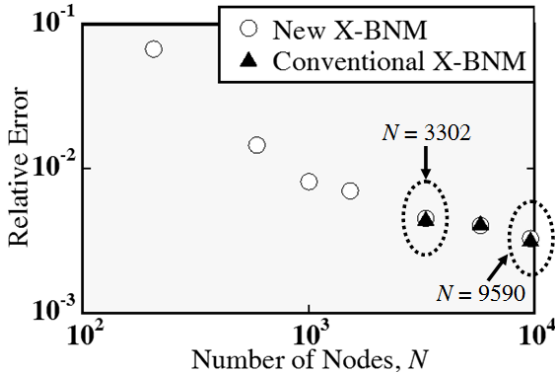

Fig. 2. Relation between the number N of nodes and the relative errors.

Let us investigate the accuracy of the new X-BNM and that of the conventional X-BNM. The relative error for the new X-BNM is determined as a function of the number N of nodes and are depicted in Fig. 2. Note that, in Fig. 2, the relative error for the conventional X-BNM is shown only for $N \ge 3302$. This is because the conventional X-BNM cannot obtain numerical solutions for $N < 3302$. Therefore stability of the new X-BNM is better than that of the conventional X-BNM. In addition, there is no obvious difference between accuracy of the new X-BNM and that of the conventional X-BNM for the case with $N \ge 3302$.

On the other hand, in $N = 9590$, computational times of new and conventional X-BNMs for determining all influence coefficients are about 252.2(s) and 210.0(s), respectively. Hence, the computational cost of the new X-BNM slightly increases as compared with that of the conventional X-BNM.

REFERENCES

[1] A. Manzin and O. Bottauscio, "Element-free Galerkin method for the analysis of electromagnetic-wave scattering," *IEEE Trans. on Magn.*, vol. 44, pp. 1366–1369, 2008.

[2] G. F. Parreira, E. J. Silva, A. Fonseca, and R. Mesquita, "The element-free Galerkin method in three-dimensional electromagnetic problems," *IEEE Trans. on Magn.*, vol. 42, pp. 711–714, 2006.

[3] M. K. Chati and S. Mukherjee, "The boundary node method for three-dimensional problems in potential theory," *Int. J. Numer. Methods Eng.*, vol. 47, pp. 1523–1547, 2000.

[4] Y. Zhang, K. R. Shao, Y. Guo, J. Zhu, D. X. Xie, and J. D. Lavers, "A comparison of point interpolative boundary meshless method based on PBF and RBF for transient eddy-current analysis," *IEEE Trans. on Magn.*, vol. 43, pp. 1497–1500, 2007.

[5] T. Itoh, A. Saitoh, A. Kamitani, and H. Nakamura, "Three dimensional extended boundary node method to potential problem," *Plasma and Fusion Research*, vol. 5, S2111, 2010.